Designing the optimal quantum cloning machine for qubit case

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Following the work of Niu and Griffiths, in *Phys.Rev.A 58, 4377(1998)*, we shall investigate the problem, how to design the optimal quantum cloning machines (QCMs) for qubit system, with the help of Bloch-sphere representation. In stead of the quality factor there, the Fiurášek's optimal condition, where the optimal cloning machine should maximize a convex mixture of the average fidelity, is used as the optimality criterion in present protocol. Almost all of the known optimal QCMs in previous works, the cloning for states with fixed polar angle, the phase-covariant cloning, the universal QCMs, the cloning for two arbitrary pure states, and the mirror phase-covariant cloning, should be discussed in a systematic way. The known results, the optimal fidelities for various input ensembles according to different optimality criteria, are recovered here. Our present scheme also offers a general way of constructing the unitary transformation to realize the optimal cloning.

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I. INTRODUCTION

One of the fundamental no-go theorems in quantum mechanics is the no-cloning theorem[1]. It is easy for us to make an arbitrary number of copies of any types of information which arrive a classic channel. However, if the information is encoded in terms of nonorthogonal quantum states, to copy or clone the information which arrives over a quantum channel is not possible without introducing errors. Instead of obtaining perfect copies, the idea of imperfect cloning was introduced by Bužek and Hillery who construct the first *quantum cloning machine* (QCM)[2]. Their work triggered an explosion in the number of investigations on quantum cloning.

The fist QCM, which was introduced by Bužek and Hillery for qubit case, is now known as the symmetric $1 \rightarrow 2$ universal quantum cloning machine (UQCM), a terms comes from the fact that it copies equally well all the pure state. The first study of non-universal or state-dependent symmetric $1 \rightarrow 2$ cloning, which is to clone at best two arbitrary pure states of a qubit, has been given by Bruß *et al.*[3]. The best-known example of state-dependent QCMs are the so-called phase-covariant QCMs, which are defined as the QCMs that copy at best states whose Bloch vector lies in the equator of the Bloch sphere [4]. As an generalization of the phase-covariant cloning, the problem of cloning to qubits, where the Euler angle θ is specified and fixed, has been introduced by Kairimipour and Rezakhani [5]. For this case, the optimal $1 \rightarrow 2$ QCMs were derived by Fiurášek [6]. Recently, Bartkiewicz *et al.* provided an optimal cloner for the qubits with known $\sin\theta$ [7]. On contrary to the symmetric $1 \rightarrow 2$ QCMs, where the outputs have the same fidelities, the asymmetric $1 \rightarrow 1 + 1$ QCMs could offer copies with different fidelity. For qubit case, Niu and Griffiths derived, in particular, the optimal asymmetric UQCM in their comprehensive study [8]. The same result was found independently by Cerf who used an algebraic approach [9,10]. A quantum circuit approach was pursued by Bužek, Hillery and Bendik [11].

Our present work originates from the open question, which has been emphasized in the review article of Scarani *et al.* [12], that there is no general result concerning state-dependent cloning and the zoology of cases *is a priori* infinite. It is an interesting task for us to find a solution for this open problem. Following the work of Niu and Griffiths in [8], we shall investigate the problem, how to design the optimal quantum cloning machines (QCMs) for qubit system, in the Bloch-sphere representation. In stead of the quality factor there, the Fiurášek's optimal condition, where the optimal cloning machine should maximize a convex mixture of the average fidelity, is used as the optimality criterion in present protocol. Almost all of the known optimal QCMs in previous works, the cloning for states with fixed polar angle, the phase-covariant cloning, the universal QCMs, the cloning for two arbitrary pure states, and the mirror phase-covariant cloning, should be discussed in a systematic way. The known results, the optimal fidelities for various input ensembles according to different optimality criteria, are recovered here. Our present scheme also offers a general way of constructing the unitary transformation to realize the optimal cloning.

Our present paper is organized as follows. Section II is a preliminary section. The argument, how to choose the basis and express the Pauli operators according to it, is given at the beginning of this section. After introducing the Bloch vector transformation for the description of QCMs, we shall develop a general scheme to find out the optimal fidelities for various input ensembles by applying the Fiurášek's optimal condition. As the two non-centered asymmetric phase-independent QCMs, the cloning for states with fixed polar angle and the phase-covariant cloning, should be discussed in Sec. III. Two centered phase-independent examples, the universal QCMs and the mirror phase-covariant cloning, can be found in Sec. IV. The phase-dependent problem, cloning two arbitrary pure states, is solved in Sec. V. Finally, some discussion about our results and a short conclusion, will be given in the last section.

II. PRELIMINARY

A. The Bloch-sphere representation

We use S to denote the set of pure qubit states to be cloned. For each $|\psi\rangle \in S$, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2+|\beta|^2 = 1$, the distribution function $q(\alpha, \beta)$ should be known, $q(\alpha, \beta) > 0$ and $\int_S q(\alpha, \beta)d\tau = 1$ with $\int_S d\tau$ for measure. At first, we introduce a density matrix decided by S,

$$\rho_S = \int_S |\psi\rangle\langle\psi|q(\alpha, \beta)d\tau, \tag{2.1}$$

and calculate its eigenvalues and eigenvectors,

$$\rho_S|\uparrow\rangle = \frac{1+\lambda}{2}|\uparrow\rangle, \rho_S|\downarrow\rangle = \frac{1-\lambda}{2}|\downarrow\rangle. \tag{2.2}$$

After introducing the new basis with its vectors as $|\uparrow\rangle$ and $|\downarrow\rangle$, we can define the identity operator and Pauli matrices,

$$\begin{aligned} \mathbf{I} = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|, \sigma_x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|, \\ \sigma_y = -i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|, \sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|, \end{aligned} \tag{2.3}$$

Now, every state (pure or mixed) ρ for qubit case, can be expressed by

$$\rho = \frac{1}{2}(\mathbf{I} + \vec{\sigma}\cdot\vec{r}), \tag{2.4}$$

where $\vec{r}$ is a three-dimensional real vectors, $\vec{r} = (r_x, r_y, r_z)$, with r_i defined by

$$r_i = \mathtt{Tr}(\sigma_i\rho), \tag{2.5}$$

Specially, the pure state $|\psi\rangle$ is described by the unit vector $\vec{n}(\theta, \phi) \equiv (n_x, n_y, n_z)$,

$$n_x = \sin\theta\cos\phi, n_y = \sin\theta\sin\phi, n_z = \cos\theta, \tag{2.6}$$

with θ and ϕ the polar and azimuthal angle in the Bloch sphere. Notice $\sum_i n_i^2 = 1$. Considering the fact that

$$|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbf{I} + \vec{\sigma}\cdot\vec{n}) \tag{2.7}$$

and

$$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}e^{-i\phi}|\downarrow\rangle, \tag{2.8}$$

is equivalent with each other, we can rewrite S, the set of states to be cloned, in the way like $S : \{\vec{n}(\theta, \phi), q(\theta, \phi)\}$, where $\int_S q(\theta, \phi)d\tau = 1$ with the measure $\int_S d\tau = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi$. Why the identity operator and Pauli operators are defined by the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ rather than the original basis $\{|0\rangle, |1\rangle\}$? The reason is that: according to Eqs. (2.3-5), the density matrix ρ_S in Eq. (2.1) can be written as

$$\rho_S = \frac{1}{2}(\mathbf{I} + \lambda\sigma_z) \tag{2.9}$$

where its components, along the $\hat{x}$ and $\hat{y}$ directions, take the value of zero,

$$\int_S \mathtt{Tr}(\sigma_j|\psi\rangle\langle\psi|)q(\theta, \phi)d\tau = 0, \tag{2.10}$$

where $j = x, y$.

Usually, we use ρ^A and ρ^B to denote the two copies of cloning, and $\vec{r^A}$ and $\vec{r^B}$ for their corresponding vectors in the Bloch-sphere representation,

$$\rho^k = \frac{1}{2}(\mathbf{I} + \vec{\sigma}\cdot\vec{r^k}), \tag{2.11}$$

where $k = A, B$. Taking $|\psi\rangle$ as the input for QCM, the *single-copy fidelity* is found to be

$$F_\psi^k = \frac{1}{2}(1 + \vec{r^k}\cdot\vec{n}) \tag{2.12}$$

which just depends on the inner product of two vector $\vec{r^k}$ and $\vec{n}$. The *average fidelity* can be defined as

$$\bar{F}^k = \int_S F_\psi^k q(\theta, \phi)d\tau. \tag{2.13}$$

Other parameters, which characterize a given set of pure states, are the so-called *averaged length* $\overline{n_i}$

$$\overline{n_i} = \int_S n_i q(\theta, \phi)d\tau, \tag{2.14}$$

and the *fluctuation* $\overline{n_i^2}$,

$$\overline{n_i^2} = \int_S n_i^2 q(\theta, \phi)d\tau, \tag{2.15}$$

where the parameters n_i with $i = x, y, z$, are defined in Eq. (2.6). In general, $(\overline{n_i})^2 \neq \overline{n_i^2}$. A frequently used relation, $\sum_i \overline{n_i^2} = 1$, can be derived from the unit condition of n_i in Eq. (2.6). Using Eq. (2.10), one may get

$$\overline{n_x} = \overline{n_y} = 0. \tag{2.16}$$

In other words, by properly choosing the operators in Eq. (2.3), we have only four parameters, $\overline{n_i^2}$ with $(i = x, y, z)$ and $\overline{n_z}$, for characterizing S to be cloned.

B. Description of the QCMs in the Bloch-sphere representation

Following the work of Niu and Griffiths [8], we shall describe the QCMs in the Bloch-sphere representation. We prepare the initial state of the system with ρ and denote the state of the environment by $\rho_{\mathtt{env}}$, the QCMs can be viewed as a unitary transformation U coupling ρ and $\rho_{\mathtt{env}}$ together,

$$\rho\otimes\rho_{\mathtt{env}} \to U\left(\rho\otimes\rho_{\mathtt{env}}\right)U^\dagger. \tag{2.17}$$

The two final copies of cloning, say, ρ^A and ρ^B, can be get by performing the partial trace over the environment, $\mathtt{Tr}_{\mathtt{env}}[U(\rho\otimes\rho_{\mathtt{env}}U^\dagger]$. Formally, $\rho^k=\sum_m E_m^k\rho(E_m^k)^\dagger$ with $\sum_m(E_m^k)^\dagger E_m^k=\mathbf{I}$ [14,15]. With $\vec{r}$ the Bloch vector for the input ρ and $\vec{r^k}$ for the outputs ρ^k, there exists a map

$$\vec{r}\to\vec{r^k}=M^k\vec{r}+\vec{\delta}^k \tag{2.18}$$

where M^k is a 3×3 real matrix, and $\vec{\delta}^k$ is a constant vector. This is an *affine map,* mapping the Bloch sphere into itself [14].

As one of the main results of present work, we shall design a series of QCMs with

$$\vec{r^k}=\begin{pmatrix}\eta_x^k & 0 & 0\\ 0 & \eta_y^k & 0\\ 0 & 0 & \eta_z^k\end{pmatrix}\begin{pmatrix}r_x\\ r_y\\ r_z\end{pmatrix}+\begin{pmatrix}0\\ 0\\ \delta_z^k\end{pmatrix} \tag{2.19}$$

where both the matrices, M^A and M^B in Eq. (2.18), are diagonal *at the same time*, while the shifts, δ_x^A, δ_y^A, δ_x^B, and δ_y^B, take a value of zero. As it is proven in appendix A1, the unitary transformation U, which takes the forms in below equations, is able to realize the vector transformation defined in Eq. (2.19),

$$\begin{aligned}U|\uparrow\rangle &\to \cos\frac{\alpha}{2}|u_+\rangle_{AB}\otimes|\uparrow\rangle_C+\sin\frac{\alpha}{2}|v_+\rangle_{AB}\otimes|\downarrow\rangle_C,\\ U|\downarrow\rangle &\to \cos\frac{\tilde{\alpha}}{2}|u_-\rangle_{AB}\otimes|\downarrow\rangle_C+\sin\frac{\tilde{\alpha}}{2}|v_-\rangle_{AB}\otimes|\uparrow\rangle_C,\end{aligned}$$

where we suppose that the copies lie in the two-dimensional space A and B after the action of U and the space C for the output state of ancilla. One may check that $U|\uparrow\rangle$ and $U|\downarrow\rangle$ are orthogonal under the condition that $\langle u_\pm|v_\pm\rangle=0$,

$$\begin{aligned}|u_+\rangle_{AB} &= \cos\frac{\beta}{2}|\uparrow\rangle_A|\uparrow\rangle_B+\sin\frac{\beta}{2}|\downarrow\rangle_A|\downarrow\rangle_B,\\ |u_-\rangle_{AB} &= \sin\frac{\tilde{\beta}}{2}|\uparrow\rangle_A|\uparrow\rangle_B+\cos\frac{\tilde{\beta}}{2}|\downarrow\rangle_A|\downarrow\rangle_B,\\ |v_+\rangle_{AB} &= \cos\frac{\gamma}{2}|\uparrow\rangle_A|\downarrow\rangle_B+\sin\frac{\gamma}{2}|\downarrow\rangle_A|\uparrow\rangle_B,\\ |v_-\rangle_{AB} &= \sin\frac{\tilde{\gamma}}{2}|\uparrow\rangle_A|\downarrow\rangle_B+\cos\frac{\tilde{\gamma}}{2}|\downarrow\rangle_A|\uparrow\rangle_B.\end{aligned} \tag{2.20}$$

We use ω_i denote one of these free parameters, $\omega_i\in\{\alpha,\tilde{\alpha},\beta,\tilde{\beta},\gamma,\tilde{\gamma}\}$. According to the calculation, which is carefully done in appendix A1, the $U(\omega)$ in Eq. (2.20) is shown to be consistent with the Bloch vector transformation in Eq. (2.19),

$$\begin{aligned}\eta_x^k &= \cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}(\cos\frac{\beta}{2}\cos\frac{\tilde{\gamma}}{2}^k+\sin\frac{\beta}{2}\sin\frac{\tilde{\gamma}}{2}^k)\\ &\quad+\sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}(\cos\frac{\tilde{\beta}}{2}\cos\frac{\gamma}{2}^k+\sin\frac{\tilde{\beta}}{2}\sin\frac{\gamma}{2}^k),\\ \eta_y^k &= \cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}(\cos\frac{\beta}{2}\cos\frac{\tilde{\gamma}}{2}^k-\sin\frac{\beta}{2}\sin\frac{\tilde{\gamma}}{2}^k)\\ &\quad+\sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}(\cos\frac{\tilde{\beta}}{2}\cos\frac{\gamma}{2}^k-\sin\frac{\tilde{\beta}}{2}\sin\frac{\gamma}{2}^k),\\ \eta_z^k &= \frac{1}{2}(\cos^2\frac{\alpha}{2}\cos\beta+\sin^2\frac{\alpha}{2}\cos\gamma^k\\ &\quad+\cos^2\frac{\tilde{\alpha}}{2}\cos\tilde{\beta}+\sin^2\frac{\tilde{\alpha}}{2}\cos\tilde{\gamma}^k),\\ \delta_z^k &= \frac{1}{2}(\cos^2\frac{\alpha}{2}\cos\beta+\sin^2\frac{\alpha}{2}\cos\gamma^k\\ &\quad-\cos^2\frac{\tilde{\alpha}}{2}\cos\tilde{\beta}-\sin^2\frac{\tilde{\alpha}}{2}\cos\tilde{\gamma}^k),\end{aligned} \tag{2.21}$$

where the the denotations,

$$\begin{aligned}\gamma^A &= \gamma,\ \tilde{\gamma}^A=\tilde{\gamma},\\ \gamma^B &= \pi-\gamma,\ \tilde{\gamma}^B=\pi-\tilde{\gamma},\end{aligned} \tag{2.22}$$

were used here for writing the results in appendix A1 into the compact form of Eq. (2.21). *It should be noted that the $U(\omega)$ of Eq. (2.20) is defined by the basis vectors, $|\uparrow\rangle$ and $|\downarrow\rangle$, in Eq. (2.2).* For cloning the set of pure states S, by joining Eqs. (2.12-16) and Eq. (2.19) together, the average fidelity in Eq. (2.13) should be

$$\bar{F}^k(\omega)=\frac{1}{2}(1+\eta_x^k\overline{n_x^2}+\eta_y^k\overline{n_y^2}+\eta_z^k\overline{n_z^2}+\delta_z^k\overline{n_z}) \tag{2.23}$$

with $k=A,B$. Both the averaged length $\overline{n_i}$ and $\overline{n_i^2}$ are just decided by S, the set of states to be cloned.

For a given case of cloning, the way of realizing the optimal fidelity may be not unique [12]. Our general unitary transformation in Eq. (2.20), as we shall show with a series of examples, offers a *sufficient and systematic* way for designing the various kinds of optimal QCMs for qubit system.

C. The optimal QCMs

In present work, we shall show that the optimal QCMs, which have been designed by different optimality criteria, for examples, the *global fidelity* in [3], the *quality factor* in [8], the *no-cloning inequality* in [9-11], *etc.*, can also be designed by the *Fiurášek's optimal condition* [6,13] where the optimal asymmetric cloning machine should maximize a convex mixture of the average fidelity $\bar{F}^A$ and $\bar{F}^B$,

$$F(\omega)=p\bar{F}^A(\omega)+(1-p)\bar{F}^B(\omega), \tag{2.24}$$

in which $p\in[0,1]$ is a parameter that controls the asymmetry of the clone. With the average fidelity in Eq.

(2.23), considering the fact that the averaged length $\overline{n_z}$ and $\overline{n_i^2}$ have been decided by S, to design the optimal QCMs is equivalent with finding out the the optimal settings of ω which satisfy the partial equations

$$\frac{\partial F(\omega)}{\partial \omega_j} \equiv p\frac{\partial \bar{F}^A(\omega)}{\partial \omega_j} + (1-p)\frac{\partial \bar{F}^B(\omega)}{\partial \omega_j} = 0. \qquad (2.25)$$

As an important case of Eq. (2.24), the optimal symmetric cloning should maximize the function,

$$F(\omega) = \frac{1}{2}(\bar{F}^A(\omega) + \bar{F}^B(\omega)), \qquad (2.26)$$

where $p = 1/2$ [6,13]. With the denotations $\eta_i = \frac{1}{2}(\eta_i^A + \eta_i^B)$ and $\delta_z = \frac{1}{2}(\delta_z^A + \delta_z^B)$, we express $F(\omega)$ in Eq. (2.26) with $F(\omega) = \frac{1}{2}(1 + \sum_i \eta_i \overline{n_i^2} + \delta_z \overline{n_z})$ and prove that the relations, $\frac{\partial F}{\partial \gamma} \propto \sin(\frac{\pi}{4} - \frac{\gamma}{2})$ and $\frac{\partial F}{\partial \tilde{\gamma}} \propto \sin(\frac{\pi}{4} - \frac{\tilde{\gamma}}{2})$, always hold without considering the actual values of $\overline{n_i^2}$ and $\overline{n_z}$ for a given S (see appendix A2). In other words, for the symmetric QCMs, the parameters γ and $\tilde{\gamma}$ are fixed,

$$\gamma = \tilde{\gamma} = \frac{\pi}{2}. \qquad (2.27)$$

Putting it back into Eq. (2.21), we find $\eta_i^A = \eta_i^B$ and $\delta_z^A = \delta_z^B$, the two copies now have the same fidelity, $F_\psi^A = F_\psi^B$, as they should be. In conclusion, the designing of symmetric QCMs can directly start from

$$F(\omega') = \frac{1}{2}(1 + \sum_i \eta_i \overline{n_i^2} + \delta_z \overline{n_z}). \qquad (2.28)$$

Under the condition in Eq. (2.27), we find the relations, $\eta_i = \eta_i^A = \eta_i^B$ and $\delta_z = \delta_z^A = \delta_z^B$, with the parameters

$$\begin{aligned}
\eta_x &= \cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos(\frac{\pi}{4} - \frac{\beta}{2}) + \sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos(\frac{\pi}{4} - \frac{\tilde{\beta}}{2}),\\
\eta_y &= \cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos(\frac{\pi}{4} + \frac{\beta}{2}) + \sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos(\frac{\pi}{4} + \frac{\tilde{\beta}}{2}),\\
\eta_z &= \frac{1}{2}(\cos^2\frac{\alpha}{2}\cos\beta + \cos^2\frac{\tilde{\alpha}}{2}\cos\tilde{\beta}),\\
\delta_z &= \frac{1}{2}(\cos^2\frac{\alpha}{2}\cos\beta - \cos^2\frac{\tilde{\alpha}}{2}\cos\tilde{\beta}). \qquad (2.29)
\end{aligned}$$

There are four variables, $\omega_j' \in \{\alpha, \tilde{\alpha}, \beta, \tilde{\beta}\}$, left here, their optimal settings should be decided by the equations $\frac{\partial F(\omega')}{\partial \omega_j'} = 0$.

D. Classification of the QCMS

Here, we make a simple classification of the QCMs according to the Bloch vector transformation in Eq. (2.19-21). Using Eq. (2.19) and Eq. (2.12), we have the single-copy fidelity,

$$\begin{aligned}
F_\psi^k &= \frac{1}{2}(1 + \eta_x^k \sin^2\theta\cos^2\phi + \eta_y^k \sin^2\theta\sin^2\phi\\
&\quad + \eta_z^k\cos^2\theta + \delta_z^k\cos\theta). \qquad (2.30)
\end{aligned}$$

A QCM is called *phase-independent* if $\eta_x^k = \eta_y^k$ because that the single-copy fidelity is now clearly independent of the phase ϕ. Usually, the phase-independent QCMs should appear in the cases with $\overline{n_x^2} = \overline{n_y^2}$. A short discussion shall be applied here to show why this happens. For the S with $\overline{n_x^2} = \overline{n_y^2}$, introducing the denotation $\eta_\perp^k = \frac{1}{2}(\eta_x^k + \eta_y^k)$, we rewrite the average fidelity as

$$\bar{F}^k = \frac{1}{2}[1 + \eta_\perp^k(1 - \overline{n_z^2}) + \eta_z^k \overline{n_z^2} + \delta_z^k \overline{n_z}]. \qquad (2.31)$$

The expression of $\eta_\perp^k$ can be get from Eq. (2.21), $\eta_\perp^k = \cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos\frac{\beta}{2}\cos\frac{\tilde{\gamma}}{2}^k + \sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos\frac{\tilde{\beta}}{2}\cos\frac{\gamma}{2}^k$. From it, $\partial\eta_\perp^k/\partial\beta = -\frac{1}{2}\cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\sin\frac{\beta}{2}\cos\frac{\tilde{\gamma}}{2}^k$ and $\partial\eta_\perp^k/\partial\tilde{\beta} = -\frac{1}{2}\sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\beta}}{2}\cos\frac{\gamma}{2}^k$. For the η_z^k and δ_z^k defined in Eq. (2.21), one may also get $\partial\eta_z^k/\partial\beta = \partial\delta_z^k/\partial\beta = -\cos^2\frac{\alpha}{2}\sin\beta$ and $\partial\eta_z^k/\partial\tilde{\beta} = -\partial\delta_z^k/\partial\tilde{\beta} = -\cos^2\frac{\tilde{\alpha}}{2}\sin\tilde{\beta}$. By joining these results together, we find the average fidelity in Eq. (2.31) with $\frac{\partial\bar{F}^k}{\partial\beta} \propto \sin\frac{\beta}{2}$ and $\frac{\partial\bar{F}^k}{\partial\tilde{\beta}} \propto \sin\frac{\tilde{\beta}}{2}$. According to the optimality equation in Eq. (2.25), the optimal settings of β and $\tilde{\beta}$ can always be chosen as

$$\beta = \tilde{\beta} = 0 \qquad (2.32)$$

for cloning the set of states with $\overline{n_x^2} = \overline{n_y^2}$. Putting $\beta = \tilde{\beta} = 0$ in Eq. (2.21), it can be seen that the transformation elements, η_x^k and η_y^k, now take a same expression, $\eta_x^k = \eta_y^k \equiv \eta_\perp^k$. Other elements, δ_z^k and η_z^k, can also be simplified by $\beta = \tilde{\beta} = 0$,

$$\begin{aligned}
\eta_\perp^k &= \cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos\frac{\tilde{\gamma}^k}{2} + \sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos\frac{\gamma^k}{2},\\
\eta_z^k &= 1 - \sin^2\frac{\alpha}{2}\sin^2\frac{\gamma^k}{2} - \sin^2\frac{\tilde{\alpha}}{2}\sin^2\frac{\tilde{\gamma}^k}{2},\\
\delta_z^k &= \sin^2\frac{\tilde{\alpha}}{2}\sin^2\frac{\tilde{\gamma}^k}{2} - \sin^2\frac{\alpha}{2}\sin^2\frac{\gamma^k}{2}. \qquad (2.33)
\end{aligned}$$

Putting $\bar{F}^k$ in Eq. (2.31) with the above parameters back to the optimal equation of Eq. (2.25), there are only four parameters, $\omega_j \in \{\alpha, \tilde{\alpha}, \gamma, \tilde{\gamma}\}$, to be decide there.

As we shall show later, almost all of the QCMs known yet are phase-independent. Certainly, a QCM is *phase-dependent* if $\eta_x^k \neq \eta_y^k$. In short, a QCM is called *centered* if $\delta_z^k = 0$, else, it is called *non-centered.*

III. NON-CENTERED PHASE-INDEPENDENT CLONING

A. The set of sates with fixed polar angle

Consider a given set of states, $|\tilde{\psi}\rangle = \cos\frac{\tilde{\theta}}{2}|\uparrow\rangle + \sin\frac{\tilde{\theta}}{2}e^{-i\phi}|\downarrow\rangle$, where the polar angle is fixed as $\tilde{\theta}$, $0 \leq \tilde{\theta} \leq \pi/2$, while the azimuthal angle is arbitrary, $\phi \in [0, 2\pi]$.

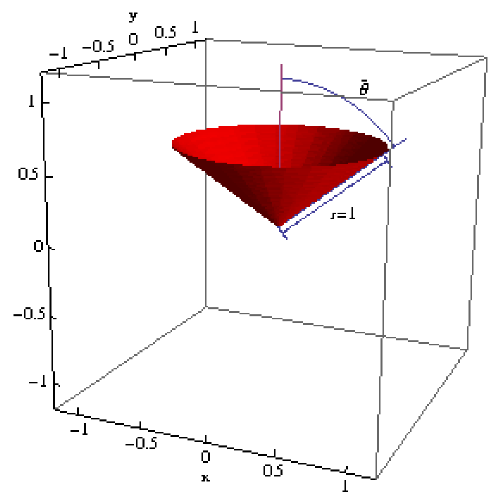

FIG. 1: The set of pure states with the Bloch vectors, $(\sin\tilde{\theta}\cos\phi, \sin\tilde{\theta}\sin\phi, \cos\tilde{\theta})$, where the polar angle has a given value $\tilde{\theta}$ while the azimuthal angle is arbitrary.

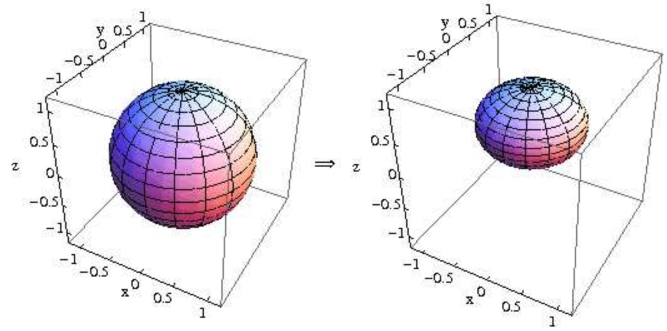

FIG. 2: The effect of the amplitude damping on the Bloch sphere, for $\vec{r} \to \left(\frac{\sqrt{2}}{2}r_x, \frac{\sqrt{2}}{2}r_y, \frac{1}{2}(1+r_z)\right)$. Note how the entire sphere shrinks to the north-pole. It is the optimal symmetric QCM for cloning the states in FIG. 1.

Its Bloch representation can be seen in FIG. 1. A simple calculation shows

$$\overline{n_z} = \cos\tilde{\theta}, \overline{n_x^2} = \overline{n_y^2} = \frac{1}{2}\sin^2\tilde{\theta}, \overline{n_z^2} = \cos^2\tilde{\theta}. \tag{3.1}$$

The optimal symmetric cloning for this case has been given in [6]. Here, its optimal asymmetric cloning is found with the settings

$$\beta = \tilde{\beta} = \alpha = 0, \tilde{\alpha} = \pi \tag{3.2}$$

according to the results in Appendix B1. The unitary transformation U takes the form,

$$\begin{aligned} U|\uparrow\rangle &\to |\uparrow\uparrow\rangle|\uparrow\rangle, \\ U|\downarrow\rangle &\to (\sin\frac{\tilde{\gamma}}{2}|\uparrow\downarrow\rangle + \cos\frac{\tilde{\gamma}}{2}|\downarrow\uparrow\rangle)|\uparrow\rangle, \end{aligned} \tag{3.3}$$

which comes from the general unitary transformation of Eq.(2.20) with the the optimal settings given at Eq. (3.2). The transformation matrix elements in Eq.(2.21) now become

$$\delta_z^k = \sin^2\frac{\tilde{\gamma}^k}{2}, \eta_\perp^k = \cos\frac{\tilde{\gamma}^k}{2}, \eta_z^k = \cos^2\frac{\tilde{\gamma}^k}{2}. \tag{3.4}$$

The average fidelities in Eq. (2.31) are also known,

$$\begin{aligned} \bar{F}^A &= \frac{1}{2}(1+\cos\frac{\tilde{\gamma}}{2}\sin^2\tilde{\theta} + \cos^2\frac{\tilde{\gamma}}{2}\cos^2\tilde{\theta} + \sin^2\frac{\tilde{\gamma}}{2}\cos\tilde{\theta}), \\ \bar{F}^B &= \frac{1}{2}(1+\sin\frac{\tilde{\gamma}}{2}\sin^2\tilde{\theta} + \sin^2\frac{\tilde{\gamma}}{2}\cos^2\tilde{\theta} + \cos^2\frac{\tilde{\gamma}}{2}\cos\tilde{\theta}). \end{aligned} \tag{3.5}$$

The parameter$\tilde{\gamma}$ in above equations should be decided by the asymmetric parameter p,

$$p = \frac{\cos\frac{\tilde{\gamma}}{2}\sin^2\tilde{\theta} + \sin\tilde{\gamma}(\cos^2\tilde{\theta} - \cos\tilde{\theta})}{(\cos\frac{\tilde{\gamma}}{2} + \sin\frac{\tilde{\gamma}}{2})\sin^2\tilde{\theta} + 2\sin\tilde{\gamma}(\cos^2\tilde{\theta} - \cos\tilde{\theta})},$$

according to the partial equation $p\partial\bar{F}^A/\partial\tilde{\gamma} + (1-p)\partial\bar{F}^B/\partial\tilde{\gamma} = 0$ in Eq. (2.25). For the symmetric case, where $p = 1/2$, the optimal setting of $\tilde{\gamma}$ is $\pi/2$ as it should be.

As an interesting result, it is found that the vector transformation defined by Eq. (3.4) is just the so-called *amplitude damping* (AD) known in [14]. Defining $\varepsilon_{\mathtt{NAD}}^k(\rho) = E_0^k\rho(E_0^k)^\dagger + E_1^k\rho(E_1^k)^\dagger$,

$$E_0^k = \begin{pmatrix} 1 & 0 \\ 0 & \cos\frac{\tilde{\gamma}^k}{2} \end{pmatrix}, E_1^k = \begin{pmatrix} 0 & \sin\frac{\tilde{\gamma}^k}{2} \\ 0 & 0 \end{pmatrix}, \tag{3.6}$$

where $\tilde{\gamma}^A = \tilde{\gamma}, \tilde{\gamma}^B = \pi - \tilde{\gamma}$, and $k = A, B$, we may verify that the two expressions, the vector transformation of Eq. (2.19) with its elements given by Eq. (3.4) and the amplitude damping defined above, are equivalent with each other. The effect of the amplitude damping on the Bloch sphere can be seen in FIG. 2 [14].

Furthermore, let $\tilde{\gamma} = \pi/2$ according to $p = 1/2$, we have the optimal symmetric fidelity from Eq. (3.5),

$$F = \frac{1}{2}[1 + \frac{\sqrt{2}}{2}\sin^2\tilde{\theta} + \frac{1}{2}(\cos^2\tilde{\theta} + |\cos\tilde{\theta}|)], \tag{3.7}$$

the result which has been given in [6].

B. The phase-covariant cloning

The so-called phase-covariant QCMs were first introduced in the problem of cloning the set of states, $|\psi\rangle = \frac{\sqrt{2}}{2}(|\uparrow\rangle + e^{-i\phi}|\downarrow\rangle)$ with ϕ taking arbitrary values [4,16]. In present work, the same problem is concerned on following two aspects: the method of deriving it by using the Fiurášek's optimal condition and its relationship with the well-known *generalized amplitude damping* (GAD) [14].

The set of the states, which lie in the equator of the Bloch sphere, are characterized by following averaged parameters,

$$\overline{n_x^2} = \overline{n_y^2} = \frac{1}{2}, \overline{n_z} = \overline{n_z^2} = 0. \tag{3.8}$$

Follow the argument in Appendix B2, the optimal settings for the free parameters in Eq. (2.22) are chosen to be,

$$\beta = \tilde{\beta} = 0, \tilde{\alpha} = \pi - \alpha, \tilde{\gamma} = \gamma, \tag{3.9}$$

according to it, the unitary transformation has the form,

$$U|\uparrow\rangle \to \cos\frac{\alpha}{2}|\uparrow\uparrow\uparrow\rangle + \sin\frac{\alpha}{2}\left(\cos\frac{\gamma}{2}|\uparrow\downarrow\rangle + \sin\frac{\gamma}{2}|\downarrow\uparrow\rangle\right)|\downarrow\rangle,$$
$$U|\downarrow\rangle \to \sin\frac{\alpha}{2}|\downarrow\downarrow\downarrow\rangle + \cos\frac{\alpha}{2}\left(\sin\frac{\gamma}{2}|\uparrow\downarrow\rangle + \cos\frac{\gamma}{2}|\downarrow\uparrow\rangle\right)|\uparrow\rangle,$$

where α takes an arbitrary value. The transformation matrix elements should be

$$\eta_{\perp}^k = \cos\frac{\gamma^k}{2}, \eta_z^k = \cos^2\frac{\gamma^k}{2}, \delta_z^k = \cos\alpha\sin^2\frac{\gamma^k}{2}. \tag{3.10}$$

Jointing it with Eq. (2.31), the average fidelity is found with

$$\bar{F}^A = \frac{1}{2}(1+\cos\frac{\gamma}{2}), \bar{F}^B = \frac{1}{2}(1+\sin\frac{\gamma}{2}). \tag{3.11}$$

One may also verify that $F_{\psi}^k = \bar{F}^k$. The result, which has been derived by Niu and Griffiths in [16], is recovered here. Certainly, the average fidelity in Eq. (3.11) can also be written in the equivalent form where p acts as the free variable. Applying the optimal equation, $p\frac{\partial \bar{F}^A}{\partial\gamma} + (1-p)\frac{\partial \bar{F}^B}{\partial\gamma} = 0$, we find

$$\cos\frac{\gamma}{2} = \frac{p}{\sqrt{(1-p)^2+p^2}}, \sin\frac{\gamma}{2} = \frac{1-p}{\sqrt{(1-p)^2+p^2}},$$

and get the fidelity,

$$\bar{F}^A = \frac{1}{2}(1+\frac{p}{\sqrt{(1-p)^2+p^2}}),$$
$$\bar{F}^B = \frac{1}{2}(1+\frac{1-p}{\sqrt{(1-p)^2+p^2}}). \tag{3.12}$$

Interestingly, we find the vector transformation with its elements in Eq. (3.10) to be the so-called *generalized amplitude damping* $\varepsilon_{\mathtt{GAD}}^k$ with operator elements [14],

$$E_0^k = \cos\frac{\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & \cos\frac{\gamma^k}{2}\end{pmatrix}, E_1^k = \cos\frac{\alpha}{2}\begin{pmatrix} 0 & \sin\frac{\gamma^k}{2} \\ 0 & 0\end{pmatrix},$$
$$E_2^k = \sin\frac{\alpha}{2}\begin{pmatrix} \cos\frac{\gamma^k}{2} & 0 \\ 0 & 1\end{pmatrix}, E_3^k = \sin\frac{\alpha}{2}\begin{pmatrix} 0 & 0 \\ \sin\frac{\gamma^k}{2} & 0\end{pmatrix},$$

where $\gamma^A = \gamma, \gamma^B = \pi - \gamma$. In FIG. 3, a vector transformation, $\vec{r} \to (\frac{\sqrt{2}}{2}r_x, \frac{\sqrt{2}}{2}r_y, \frac{1}{4}+\frac{1}{2}r_z)$, is depicted as an example for $\varepsilon_{\mathtt{GAD}}$.

IV. CENTERED PHASE-INDEPENDENT CLONING

A. The universal cloning

A QCM is called *universal* if it copies equally well all the pure states $|\psi\rangle$ distributed in the surface of the Bloch sphere with equal probability. This problem can be characterized by

$$\overline{n_i^2} = \frac{1}{3}, \overline{n_z} = 0 \tag{4.1}$$

with $i = x, y, z$. According to the calculation in appendix B3, we find the optimal settings,

$$\tilde{\gamma} = \gamma, \beta = \tilde{\beta} = 0, \tilde{\alpha} = \alpha, \tag{4.2}$$

where α and γ are decided by p,

$$\cos\frac{\alpha}{2} = \frac{1}{\sqrt{2(1-p+p^2}}, \sin\frac{\alpha}{2} = \frac{\sqrt{1-2p+p^2}}{\sqrt{2(1-p+p^2)}}$$
$$\cos\frac{\gamma}{2} = \frac{p}{\sqrt{2p^2-2p+1}}, \sin\frac{\gamma}{2} = \frac{1-p}{\sqrt{2p^2-2p+1}}, \tag{4.3}$$

The unitary transformation, with the above optimal settings, is known

$$U|\uparrow\rangle \to \cos\frac{\alpha}{2}|\uparrow\uparrow\uparrow\rangle + \sin\frac{\alpha}{2}\left(\cos\frac{\gamma}{2}|\uparrow\downarrow\rangle + \sin\frac{\gamma}{2}|\downarrow\uparrow\rangle\right)|\downarrow\rangle,$$
$$U|\downarrow\rangle \to \cos\frac{\alpha}{2}|\downarrow\downarrow\downarrow\rangle + \sin\frac{\alpha}{2}\left(\sin\frac{\gamma}{2}|\uparrow\downarrow\rangle + \cos\frac{\gamma}{2}|\downarrow\uparrow\rangle\right)|\uparrow\rangle.$$

Its transformation elements should be

$$\eta_i^A = \frac{p}{1-p+p^2}, \eta_i^B = \frac{1-p}{1-p+p^2}, \tag{4.4}$$

while $\delta_z^k = 0$. From Eq. (2.30), the single-copy fidelities should be

$$F_{\psi}^A = \frac{1}{2}(1+\frac{p}{1-p+p^2}), F_{\psi}^B = \frac{1}{2}(1+\frac{1-p}{1-p+p^2}), \tag{4.5}$$

which saturate the no-cloning inequality

$$\sqrt{(1-F_{\psi}^A)(1-F_{\psi}^B)} \geq \frac{1}{2} - (1-F_{\psi}^A) - (1-F_{\psi}^B).$$

Certainly, $\bar{F}^k = F_{\psi}^k$ since the equal probability for each $|\psi\rangle$. The special transformation of Eq. (4.4), is already known to be the *depolarizing channel* [8,10]

B. Centered symmetric phase-independent cloning

For the set of states with

$$\overline{n_x^2} = \overline{n_y^2}, \overline{n_z^2} > 0, \overline{n_z} = 0, \tag{4.6}$$

the optimal symmetric QCMs can be easily designed because there are four parameters already decided,

$$\gamma = \tilde{\gamma} = \frac{\pi}{2}, \beta = \tilde{\beta} = 0 \tag{4.7}$$

according to Eq. (2.27) and Eq. (2.32), respectively. As it is calculated in appendix B4, the other two variables,

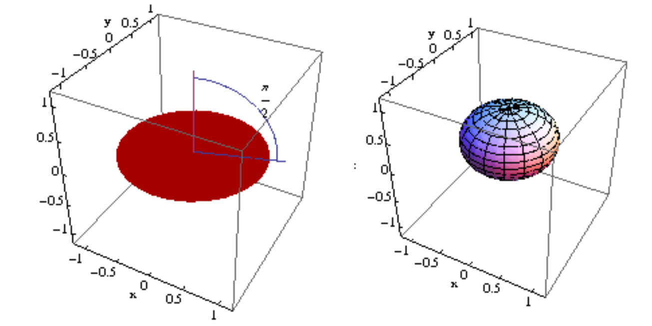

FIG. 3: On the left is the set of pure states in the equator of the Bloch sphere. The generalized amplitude damping in Eq. (3.4), which is the optimal QCM for the states on left, is chosen with $\vec{r} \to (\frac{\sqrt{2}}{2}r_x, \frac{\sqrt{2}}{2}r_y, \frac{1}{4}+\frac{1}{2}r_z)$. In fact, its center can arrange from $(0,0,\frac{1}{2})$ to $(0,0,-\frac{1}{2})$.

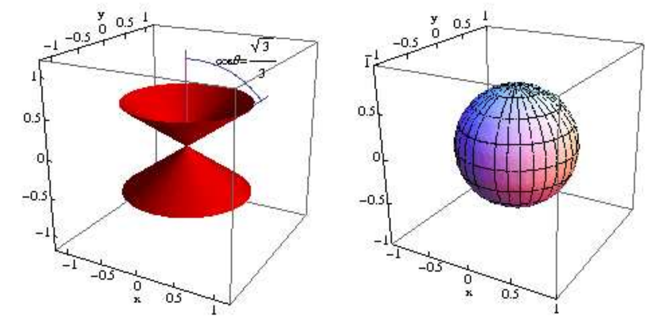

FIG. 4: On left is a set of states with $\sin\theta = \sqrt{\frac{2}{3}}$ while ϕ taking an arbitrary value. It is a special case of the so-called mirror phase-covariant cloning with its optimal symmetric cloning machine, which is on the right, to be $\vec{r} \to \frac{2}{3}\vec{r}$. All the Bloch vectors shrink with a factor of $\frac{2}{3}$.

α and $\tilde{\alpha}$, should be

$$\begin{aligned} \tilde{\alpha} &= \alpha, \\ \cos\alpha &= \frac{\overline{n_z^2}}{\sqrt{(\overline{n_z^2})^2 + 2(1-\overline{n_z^2})^2}}, \\ \sin\alpha &= \frac{\sqrt{2}(1-\overline{n_z^2})}{\sqrt{(\overline{n_z^2})^2 + 2(1-\overline{n_z^2})^2}}. \end{aligned} \tag{4.8}$$

The unitary transformation with the optimal settings above is

$$\begin{aligned} U|\uparrow\rangle &\to \cos\frac{\alpha}{2}|\uparrow\uparrow\uparrow\rangle + \frac{\sqrt{2}}{2}\sin\frac{\alpha}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)|\downarrow\rangle, \\ U|\downarrow\rangle &\to \cos\frac{\alpha}{2}|\downarrow\downarrow\downarrow\rangle + \frac{\sqrt{2}}{2}\sin\frac{\alpha}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)|\uparrow\rangle, \end{aligned}$$

Now, the diagonal elements of the transformation matrix, which hold for both the copies, are found to be,

$$\eta_x = \eta_y = \frac{\sqrt{2}}{2}\sin\alpha, \eta_z = \frac{1}{2}(1+\cos\alpha), \tag{4.9}$$

while $\delta_z = 0$. Using Eq. (2.28), the optimal symmetric, which is for the case defined in Eq. (4.6), is derived out,

$$F = \frac{1}{2} + \frac{1}{4}\left(\overline{n_z^2} + \sqrt{(\overline{n_z^2})^2 + 2(1-\overline{n_z^2})^2}\right), \tag{4.10}$$

The Bloch vector transformation in Eq. (4.9) can be verified to be equivalent with the so-called *symmetric Pauli channel* (SP), $\varepsilon_{\mathtt{SP}}(\rho) = \sum_m E_m \rho E_m^\dagger$, with the operation elements,

$$\begin{aligned} E_0 = \sqrt{1-2a^2-b^2}\mathbf{I}, E_1 = a\sigma_x \\ E_2 = a\sigma_y, E_3 = b\sigma_z, \end{aligned} \tag{4.11}$$

with $a = \frac{1}{2}\sin\frac{\alpha}{2}$ and $b = \frac{\sqrt{2}}{2}(\cos\frac{\alpha}{2} - \frac{\sqrt{2}}{2}\sin\frac{\alpha}{2})$.

Example 1. The optimal universal symmetric QCM. Considering the special case, $\overline{n_i^2} = 1/3$ and $\overline{n_z} = 0$, which appears in cloning all the unit Bloch vectors with equal probability. With $\sin\frac{\alpha}{2} = \sqrt{1/3}$ and $\cos\frac{\alpha}{2} = \sqrt{2/3}$ from Eq. (4.8), the unitary transformation now takes the form $U|\uparrow\rangle \to \sqrt{2/3}|\uparrow\uparrow\rangle|\uparrow\rangle + \sqrt{1/3}|\Psi^+\rangle|\downarrow\rangle$ and $U|\downarrow\rangle \to \sqrt{2/3}|\downarrow\downarrow\rangle|\downarrow\rangle + \sqrt{1/3}|\Psi^+\rangle|\uparrow\rangle$ where $|\Psi^+\rangle = \sqrt{1/2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$. From Eq. (4.10), the optimal fidelity is $F = 5/6$. The well-known result given by Bǔzek and Hillery [2], is recovered here. Certainly, the same result can also be given by the asymmetric universal QCM in Eq. (4.5) if p is set with $1/2$ there. One may verify that $a = b = \sqrt{1/6}$ from Eq. (4.11) and the symmetric Pauli Channel becomes the depolarizing channel.

Example 2. Optimal mirror phase-covariant cloning. Recently, Bartkiewicz *et al*. proposed a quantum cloning machine, which clones a qubit into two copies assuming known modulus of expectation value of Pauli σ_z matrix [7]. This is generalized version of Fiurášek original one of cloning the set of states with fixed value of θ [6]. The so-called *mirror phase-covariant cloning* (MPC) can be rephrased as to clone the states $|\tilde{\psi}\rangle = \cos\frac{\tilde{\theta}}{2}|\uparrow\rangle + \sin\frac{\tilde{\theta}}{2}e^{-i\phi}|\downarrow\rangle$ with the polar angle takes one of the values, $\tilde{\theta}$ and $\pi - \tilde{\theta}$, with an equal probability while ϕ has an arbitrary value in the domain $[0, 2\pi]$. This set of states is characterized by the averaged length and fluctuations, $\overline{n_x^2} = \overline{n_y^2} = \frac{1}{2}\sin^2\tilde{\theta}, \overline{n_z^2} = \cos^2\tilde{\theta}, \overline{n_z} = 0$. With these values in hands, we have the optimal setting, $\sin\alpha = \frac{\sqrt{2}\sin^2\tilde{\theta}}{\sqrt{\cos^4\tilde{\theta}+2\sin^4\tilde{\theta}}}$,$\cos\alpha = \frac{\cos^2\tilde{\theta}}{\sqrt{\cos^4\tilde{\theta}+2\sin^4\tilde{\theta}}}$ from Eq. (4.8) and the optimal fidelity

$$F = \frac{1}{2} + \frac{1}{4}\left(\cos^2\tilde{\theta} + \sqrt{\cos^4\tilde{\theta} + 2\sin^4\tilde{\theta}}\right).$$

according to Eq. (4.10). An example of the mirror phase-covariant with $\cos\theta = 1/\sqrt{3}$ is depicted in Fig. 4 where $\vec{r} \to 2\vec{r}/3$. This example comes from the symmetric Pauli channel in Eq. (4.11) with $a = b = 1/\sqrt{6}$ there.

V. SYMMETRIC PHASE-DEPENDENT CLONING

In above sections, several types of phase-independent cloning have been discussed. Usually, if $\eta_x^k \neq \eta_y^k$, the single-copy fidelity should depend on the actual value of ϕ. Here, we shall discuss the symmetric phase-dependent QCMs for case with

$$\overline{n_x^2} \neq \overline{n_y^2} = 0. \tag{5.1}$$

As it is calculated in appendix B5, the optimal setting for such case is

$$\gamma = \tilde{\gamma} = \frac{\pi}{2}, \alpha = 0, \tilde{\alpha} = \pi,$$
$$\sin(\frac{\pi}{4} - \frac{\beta}{2}) = \frac{-\overline{n_x^2} + \sqrt{(\overline{n_x^2})^2 + 8(\overline{n_z^2} + \overline{n_z})^2}}{4(\overline{n_z^2} + \overline{n_z})}, \tag{5.2}$$

while $\tilde{\beta}$ takes an arbitrary value. This setting shall simplify the unitary transformation in Eq. (2.20) into

$$U|\uparrow\rangle \rightarrow (\cos\frac{\beta}{2}|\uparrow\uparrow\rangle + \sin\frac{\beta}{2}|\downarrow\downarrow\rangle)|\uparrow\rangle,$$
$$U|\downarrow\rangle \rightarrow \sqrt{\frac{1}{2}}\,(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\,|\uparrow\rangle. \tag{5.3}$$

The vector transformation with its elements in Eq. (2.29) now take the simple forms,

$$\eta_x = \cos(\frac{\pi}{4} - \frac{\beta}{2}), \eta_y = \cos(\frac{\pi}{4} + \frac{\beta}{2}), \eta_z = \delta_z = \frac{1}{2}\cos\beta. \tag{5.4}$$

Jointing Eq. (5.4) and Eq. (2.28) together, we find the optimal symmetric fidelity

$$F = \frac{1}{2}\{1 + \cos(\frac{\pi}{4} - \frac{\beta}{2})[\overline{n_x^2} + \sin(\frac{\pi}{4} - \frac{\beta}{2})(\overline{n_z^2} + \overline{n_z})]\}. \tag{5.5}$$

In the operator-sum operation representation, the vector transformation with its elements in Eq. (5.4) is called the *deformed amplitude damping*, $\varepsilon_{\mathtt{DAD}}$, with its elements as

$$E_0 = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ \sin\frac{\beta}{2} & 0 \end{pmatrix}, E_1 = \begin{pmatrix} \cos\frac{\beta}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}. \tag{5.6}$$

The term DAD comes from the fact that $\varepsilon_{\mathtt{DAD}}$ will reduce to the amplitude damping in Eq. (3.6) if $\beta = 0$.

Example 1. Cloning two arbitrary pure states. The first study of state-dependent cloning is to perform symmetric cloning of two states [3], say, $|\psi_1\rangle = \cos\frac{\tilde{\theta}}{2}|\uparrow\rangle + \sin\frac{\tilde{\theta}}{2}|\downarrow\rangle$ and $|\psi_2\rangle = \cos\frac{\tilde{\theta}}{2}|\uparrow\rangle - \sin\frac{\tilde{\theta}}{2}|\downarrow\rangle$, with equal probability 1/2. These states lie in the $x-z$ plane of the Bloch sphere. Using s to denote the overlap of the states, $s = \cos\tilde{\theta} = \langle\psi_1|\psi_2\rangle$, with $\overline{n_x^2} = 1 - s^2, \overline{n_y^2} = 0, \overline{n_z^2} = s^2, \overline{n_z} = s$, we find the optimal setting of β in Eq. (5.2), $\sin(\frac{\pi}{4} - \frac{\beta}{2}) = \frac{1}{4s}\left(-1 + s + \sqrt{1 - 2s + 9s^2}\right)$, and express the fidelity in Eq. (5.5) in the way like

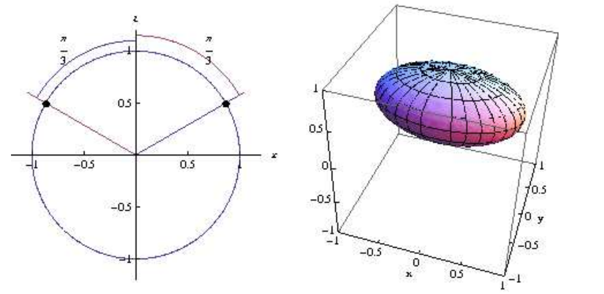

FIG. 5: On the left, two unit vectors with the relative angle $\frac{2\pi}{3}$ are used to denote the two pure states with their overlap to be $\frac{1}{2}$. The optimal symmetric QCM is found with the vector transformation, the so-called deformed amplitude damping, on the right where $\vec{r} \rightarrow \left(\frac{\sqrt{3}}{2}r_x, \frac{1}{2}r_y, \frac{\sqrt{3}}{4}(1 + r_z)\right)$. Note that the rotation-invariance along the $\hat{z}$ axe, which usually appears in the phase-independent cloning, disappears here.

$$F = \frac{1}{2} + \frac{\sqrt{2}}{32s}(1 + s)(3 - 3s + \sqrt{1 - 2s + 9s^2})$$
$$\times\sqrt{-1 + 2s + 3s^2 + (1 - s)\sqrt{1 - 2s + 9s^2}}.$$

It is the exact result which has been given in [3]. As an example, the optimal symmetric cloning for two states with $s = \frac{1}{2}$ is depicted in Fig. 5.

Example 2. Cloning two states with different probabilities. Consider a set containing just two states, $|\psi_1\rangle$ and $|\psi_2\rangle$, with fixed overlap $\langle\psi_1|\psi_2\rangle = \frac{1}{2}$. Suppose the probability of $|\psi_1\rangle$ is denoted by k, $0 \leq k \leq \frac{1}{2}$, and $|\psi_2\rangle$ with a probability of $1 - k$. Following the argument in Sec. II, we chose a Bloch-sphere representation where the Bloch vector of ρ, $\rho = k|\psi_1\rangle\langle\psi_1| + (1 - k)|\psi_2\rangle\langle\psi_2|$, points along the direction of $\hat{z}$. The two states are specified by their corresponding Bloch vectors, $\vec{r}_1 = (\frac{\sqrt{3}(1-k)}{2\sqrt{1-3k+3k^2}}, 0, \frac{3k-1}{2\sqrt{1-3k+3k^2}})$ and $\vec{r}_2 = (\frac{-\sqrt{3}k}{2\sqrt{1-3k+3k^2}}, 0, \frac{2-3k}{2\sqrt{1-3k+3k^2}})$, with the relative angle to be $\frac{2\pi}{3}$. One may verify that $\overline{n_z^2} = kr_{1z}^2 + (1-k)r_{2z}^2$, $\overline{n_z} = kr_{1z} + (1-k)r_{2z}$, $\overline{n_x^2} = kr_{1x}^2 + (1-k)r_{2x}^2$ and $\overline{n_y^2} = 0$. The fidelities, F_{ψ_i} for each state $|\psi_i\rangle$ and the average fidelity $\bar{F}$, are found with $F_{\psi_i} = \frac{1}{2}\{1 + \cos(\frac{\pi}{4} - \frac{\beta}{2})[r_{ix}^2 + \sin(\frac{\pi}{4} - \frac{\beta}{2})(r_{iz}^2 + r_{iz})]\}$ and $F = \frac{1}{2}\{1 + \cos(\frac{\pi}{4} - \frac{\beta}{2})[\overline{n_x^2} + \sin(\frac{\pi}{4} - \frac{\beta}{2})(\overline{n_z^2} + \overline{n_z})]\}$. For the case with $k = \frac{1}{2}$, $\sin(\frac{\pi}{4} - \frac{\beta}{2}) = \frac{1}{2}$, we find that $F_{\psi_1} = F_{\psi_2} = F = \frac{1}{2}(1 + \frac{9\sqrt{3}}{16})$, which can also be derived from Eq. (5.9) by letting $s = \frac{1}{2}$ there. In general, if the probabilities k and $1 - k$ are unequal, the state with the larger probability should have the higher fidelity to be cloned. This can be seen from the extremal case with $k \rightarrow 0$. With the optimal setting $\sin(\frac{\pi}{4} - \frac{\beta}{2}) \rightarrow \frac{\sqrt{2}}{2}$ get from Eq. (5.2). The single-copy fidelities, $F_{\psi_1} \rightarrow \frac{7+3\sqrt{2}}{16}$ and $F_{\psi_2} \rightarrow 1$, are different. The average fidelity F, $F = kF_{\psi_1} + (1-k)F_{\psi_2}$, approaches 1.

VI. DISCUSSION

The asymmetric phase-covariant cloning plays an important role in the BB84 protocol [17-19]. In Sec. III, the vector transformation for such case is shown to be the well-known *generalized amplitude damping*. In principle, it can be detected by using the approach of *quantum process tomography* [14] or the optimal estimation scheme developed in [20].

In conclusion, with the help of the Bloch vector transformation, we developed a scheme to design the optimal QCMs according to the Fiurášek's optimal condition. Our protocol is shown to be successful in recovering the known optimal fidelities for various input ensembles, and it should represents a general solution for the problem of optimal cloning in qubit system.

Acknowledgments

We would like to thank Prof. Lu Xiaofu for discussion which helps us a lot.

Appendix A: Some proofs for section II

1. The proof for equation (2.21)

The unitary transformation in Eq. (2.20), which makes the calculation of the average fidelity in Eq. (2.23) with a simple form, is rather special in the sense that both the matrices M^A and M^B are diagonal at the same time, a result which has not been shown before. There are sufficient reasons for us to give the derivation for Eq. (2.21) in detail and to make sure that there is no assumption needed here.

Our proof is given for the general mixed state rather than the pure one. For an arbitrary state, $\rho=\frac{1}{2}(\mathbf{I}+\vec{\sigma}\cdot\vec{r})$ with r for its length and $\vec{n}$ in Eq. (2.6) for its direction, we can write it with an equivalent form, $\rho=\frac{1+r}{2}|\psi\rangle\langle\psi|+\frac{1-r}{2}|\psi^{\perp}\rangle\langle\psi^{\perp}|$, with $|\psi\rangle=\cos\frac{\theta}{2}|\uparrow\rangle+\sin\frac{\theta}{2}e^{-i\phi}|\downarrow\rangle$ and $|\psi^{\perp}\rangle=\sin\frac{\theta}{2}|\uparrow\rangle-\cos\frac{\theta}{2}e^{-i\phi}|\downarrow\rangle$. Using $|\Psi\rangle$ and $|\Psi^{\perp}\rangle$ to denote $U|\psi\rangle$ and $U|\psi^{\perp}\rangle$ respectively, we shall do the calculations, $\rho^A=\frac{1+r}{2}\mathtt{Tr}_{BC}|\Psi\rangle\langle\Psi|)+\frac{1-r}{2}\mathtt{Tr}_{BC}|\Psi^{\perp}\rangle\langle\Psi^{\perp}|$ and $\rho^B=\frac{1+r}{2}\mathtt{Tr}_{AC}|\Psi\rangle\langle\Psi|)+\frac{1-r}{2}\mathtt{Tr}_{AC}|\Psi^{\perp}\rangle\langle\Psi^{\perp}|$. After writing ρ^k as $\rho^k=\frac{1}{2}(\mathbf{I}+\vec{\sigma}\cdot\vec{r^k})$, we shall prove that the two vectors, $\vec{r^k}$ for the $k-th$ copy and $\vec{r}$ for input, should be related by the way in Eq. (2.19).

Let's calculate $Tr_{BC}|\Psi\rangle\langle\Psi|$ at first. According to the definition of $|\psi\rangle$ and the unitary transformation in Eq. (2.20), we write $|\Psi\rangle$ as

$$
\begin{aligned}
|\Psi\rangle=&(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\theta}{2}|\uparrow_A\rangle+\sin\frac{\tilde{\alpha}}{2}\cos\frac{\tilde{\gamma}}{2}\sin\frac{\theta}{2}e^{-i\phi}|\downarrow_A\rangle)\otimes|\uparrow\rangle_B\rangle\otimes|\uparrow_C\rangle,\\
&+(\sin\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\gamma}}{2}\sin\frac{\theta}{2}e^{-i\phi}|\uparrow_A\rangle+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\theta}{2}|\downarrow_A\rangle)\otimes|\downarrow_B\rangle\otimes|\uparrow_C\rangle,\\
&+(\cos\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\beta}}{2}\sin\frac{\theta}{2}e^{-i\phi}|\uparrow_A\rangle+\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\cos\frac{\theta}{2}|\downarrow_A\rangle)\otimes|\uparrow_B\rangle\otimes|\downarrow_C\rangle,\\
&+(\sin\frac{\alpha}{2}\cos\frac{\gamma}{2}\cos\frac{\theta}{2}|\uparrow_A\rangle+\cos\frac{\tilde{\alpha}}{2}\cos\frac{\tilde{\beta}}{2}\sin\frac{\theta}{2}e^{-i\phi}|\downarrow_A\rangle)\otimes|\downarrow_B\rangle\otimes|\downarrow_C\rangle.
\end{aligned}
$$

After performing the operation of partial trace, we shall get a density matrix, $\begin{pmatrix} a_{11} & a_{12}\\ a_{12}^* & a_{22}\end{pmatrix}=\mathtt{Tr}_{BC}|\Psi\rangle\langle\Psi|$, with

$$
\begin{aligned}
a_{11}&=(\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}+\sin^2\frac{\alpha}{2}\cos^2\frac{\gamma}{2})\cos^2\frac{\theta}{2}+(\sin^2\frac{\tilde{\alpha}}{2}\sin^2\frac{\tilde{\gamma}}{2}+\cos^2\frac{\tilde{\alpha}}{2}\sin^2\frac{\tilde{\beta}}{2})\sin^2\frac{\theta}{2},\\
a_{22}&=(\cos^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}+\sin^2\frac{\alpha}{2}\sin^2\frac{\gamma}{2})\cos^2\frac{\theta}{2}+(\sin^2\frac{\tilde{\alpha}}{2}\cos^2\frac{\tilde{\gamma}}{2}+\cos^2\frac{\tilde{\alpha}}{2}\cos^2\frac{\tilde{\beta}}{2})\sin^2\frac{\theta}{2},\\
a_{12}&=\frac{1}{2}\sin\theta[e^{i\phi}(\cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos\frac{\beta}{2}\cos\frac{\tilde{\gamma}}{2}+\sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos\frac{\tilde{\beta}}{2}\cos\frac{\gamma}{2}\\
&\quad+e^{-i\phi}(\cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\sin\frac{\beta}{2}\sin\frac{\tilde{\gamma}}{2}+\sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\beta}}{2}\sin\frac{\gamma}{2}].
\end{aligned}
$$

Noting that the Bloch vector of $|\psi^{\perp}\rangle\langle\psi^{\perp}|$ is $-\vec{n}$ and the vector for $|\psi\rangle\langle\psi|$ is $\vec{n}$, $\mathtt{Tr}_{BC}|\Psi^{\perp}\rangle\langle\Psi^{\perp}|$ can be calculated by substituting $\pi-\theta$ and $\pi+\phi$ for θ and ϕ in $\mathtt{Tr}_{BC}|\Psi\rangle\langle\Psi|$, respectively. Denoting $\rho^A=\begin{pmatrix}\rho_{11}^A & \rho_{12}^A\\ \rho_{21}^A & \rho_{22}^B\end{pmatrix}=\frac{1+r}{2}|\Psi\rangle\langle\Psi|+$

$\frac{1-r}{2}|\Psi^{\perp}\rangle\langle\Psi^{\perp}|$, the matrix elements should be,

$$\rho^A_{11} = \frac{1+r\cos\theta}{2}[\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}+\sin^2\frac{\alpha}{2}\cos^2\frac{\gamma}{2}] + \frac{1-r\cos\theta}{2}[\sin^2\frac{\tilde\alpha}{2}\sin^2\frac{\tilde\gamma}{2}+\cos^2\frac{\tilde\alpha}{2}\sin^2\frac{\tilde\beta}{2}],$$
$$\rho^A_{22} = \frac{1+r\cos\theta}{2}[\cos^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}+\sin^2\frac{\alpha}{2}\sin^2\frac{\gamma}{2}] + \frac{1-r\cos\theta}{2}[\sin^2\frac{\tilde\alpha}{2}\cos^2\frac{\tilde\gamma}{2}+\cos^2\frac{\tilde\alpha}{2}\cos^2\frac{\tilde\beta}{2}],$$

while $\rho^A_{12} = ra_{12}$ and $\rho^A_{21} = (\rho^A_{12})^*$. Defining $\rho^A = \frac{1}{2}(\mathbf{I}+\vec\sigma\cdot\vec r^A)$ with $r^A_i = \mathtt{Tr}(\sigma_i\rho^A)$, we find $r^A_x = \eta^A_x r\sin\theta\cos\phi$, $r^A_y = \eta^A_y r\sin\theta\sin\phi$, and $r^A_z = \delta^A_z + \eta^A_z r\cos\theta$, where the parameters, η^A_i and δ^A_z, take the form

$$\eta^A_x = \cos\frac{\alpha}{2}\sin\frac{\tilde\alpha}{2}(\cos\frac{\beta}{2}\cos\frac{\tilde\gamma}{2}+\sin\frac{\beta}{2}\sin\frac{\tilde\gamma}{2})+\sin\frac{\alpha}{2}\cos\frac{\tilde\alpha}{2}(\cos\frac{\tilde\beta}{2}\cos\frac{\gamma}{2}+\sin\frac{\tilde\beta}{2}\sin\frac{\gamma}{2}),$$
$$\eta^A_y = \cos\frac{\alpha}{2}\sin\frac{\tilde\alpha}{2}(\cos\frac{\beta}{2}\cos\frac{\tilde\gamma}{2}-\sin\frac{\beta}{2}\sin\frac{\tilde\gamma}{2})+\sin\frac{\alpha}{2}\cos\frac{\tilde\alpha}{2}(\cos\frac{\tilde\beta}{2}\cos\frac{\gamma}{2}-\sin\frac{\tilde\beta}{2}\sin\frac{\gamma}{2}),$$
$$\eta^A_z = \frac{1}{2}[\cos^2\frac{\alpha}{2}\cos\beta+\sin^2\frac{\alpha}{2}\cos\gamma+\cos^2\frac{\tilde\alpha}{2}\cos\tilde\beta+\sin^2\frac{\tilde\alpha}{2}\cos\tilde\gamma],$$
$$\delta^A_z = \frac{1}{2}[\cos^2\frac{\alpha}{2}\cos\beta+\sin^2\frac{\alpha}{2}\cos\gamma-\cos^2\frac{\tilde\alpha}{2}\cos\tilde\beta-\sin^2\frac{\tilde\alpha}{2}\cos\tilde\gamma].$$

Recalling the fact that the initial state ρ is characterized by the Bloch vector $\vec r = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the vectors, $\vec r^A$ and $\vec r$, can be proved to satisfy the vector transformation in Eq. (2.19). The reduced density matrix ρ^B, $\rho^B = \frac{1+r}{2}\mathtt{Tr}_{AC}|\Psi\rangle\langle\Psi| + \frac{1-r}{2}\mathtt{Tr}_{AC}|\Psi^{\perp}\rangle\langle\Psi^{\perp}|$, can also be calculated in a similar way. The state $U|\psi\rangle$ with the unitary transformation U in Eq. (2.20) can be rewritten as

$$|\Psi\rangle = (\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\theta}{2}|\uparrow_B\rangle + \sin\frac{\tilde\alpha}{2}\sin\frac{\tilde\gamma}{2}\sin\frac{\theta}{2}e^{-i\phi}|\downarrow_B\rangle)\otimes|\uparrow_A\rangle\otimes|\uparrow_C\rangle,$$
$$= (\sin\frac{\tilde\alpha}{2}\cos\frac{\tilde\gamma}{2}\sin\frac{\theta}{2}e^{-i\phi}|\uparrow_B\rangle + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\theta}{2}|\downarrow_B\rangle)\otimes|\downarrow_A\rangle\otimes|\uparrow_C\rangle,$$
$$= (\cos\frac{\tilde\alpha}{2}\sin\frac{\tilde\beta}{2}\sin\frac{\theta}{2}e^{-i\phi}|\uparrow_B\rangle + \sin\frac{\alpha}{2}\cos\frac{\gamma}{2}\cos\frac{\theta}{2}|\downarrow_B\rangle)\otimes|\uparrow_A\rangle\otimes|\downarrow_C\rangle,$$
$$= (\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\cos\frac{\theta}{2}|\uparrow_B\rangle + \cos\frac{\tilde\alpha}{2}\cos\frac{\tilde\beta}{2}\sin\frac{\theta}{2}e^{-i\phi}|\downarrow_B\rangle)\otimes|\downarrow_A\rangle\otimes|\downarrow_C\rangle,$$

which is convenient for performing $\mathtt{Tr}_{AC}$. Let $\begin{pmatrix} b_{11} & b_{12} \\ b^*_{12} & b_{22}\end{pmatrix} = \mathtt{Tr}_{AC}|\Psi\rangle\langle\Psi|$, we shall get

$$b_{11} = (\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}+\sin^2\frac{\alpha}{2}\sin^2\frac{\gamma}{2})\cos^2\frac{\theta}{2} + (\sin^2\frac{\tilde\alpha}{2}\cos^2\frac{\tilde\gamma}{2}+\cos^2\frac{\tilde\alpha}{2}\sin^2\frac{\tilde\beta}{2})\sin^2\frac{\theta}{2},$$
$$b_{22} = (\cos^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}+\sin^2\frac{\alpha}{2}\cos^2\frac{\gamma}{2})\cos^2\frac{\theta}{2} + (\sin^2\frac{\tilde\alpha}{2}\sin^2\frac{\tilde\gamma}{2}+\cos^2\frac{\tilde\alpha}{2}\cos^2\frac{\tilde\beta}{2})\sin^2\frac{\theta}{2},$$
$$b_{12} = \frac{1}{2}\sin\theta[e^{i\phi}(\cos\frac{\alpha}{2}\sin\frac{\tilde\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\tilde\gamma}{2}+\sin\frac{\alpha}{2}\cos\frac{\tilde\alpha}{2}\cos\frac{\tilde\beta}{2}\sin\frac{\gamma}{2})$$
$$+e^{-i\phi}(\cos\frac{\alpha}{2}\sin\frac{\tilde\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\tilde\gamma}{2}+\sin\frac{\alpha}{2}\cos\frac{\tilde\alpha}{2}\sin\frac{\tilde\beta}{2}\cos\frac{\gamma}{2})].$$

The reduced density matrix, $\mathtt{Tr}_{AC}|\Psi^{\perp}\rangle\langle\Psi^{\perp}|$, can also be derived out by substituting $\pi-\theta$ and $\pi+\phi$ for the angles θ and ϕ in $\mathtt{Tr}_{AC}|\Psi\rangle\langle\Psi|$ given above. Denote $\vec r^B$ the Bloch vector for ρ^B, $\rho^B = \frac{1+r}{2}\mathtt{Tr}_{AC}|\Psi\rangle\langle\Psi| + \frac{1-r}{2}\mathtt{Tr}_{AC}|\Psi^{\perp}\rangle\langle\Psi^{\perp}|$, and $r^B_i = \mathtt{Tr}(\sigma_i\rho^B)$, we shall get the results, $r^B_x = \eta^B_x r\sin\theta\cos\phi$, $r^B_y = \eta^B_y r\sin\theta\sin\phi$, and $r^B_z = \eta^B_z r\cos\theta + \delta^B_z$, where

$$\eta^B_x = \cos\frac{\alpha}{2}\sin\frac{\tilde\alpha}{2}(\cos\frac{\beta}{2}\sin\frac{\tilde\gamma}{2}+\sin\frac{\beta}{2}\cos\frac{\tilde\gamma}{2})+\sin\frac{\alpha}{2}\cos\frac{\tilde\alpha}{2}(\cos\frac{\tilde\beta}{2}\sin\frac{\gamma}{2}+\sin\frac{\tilde\beta}{2}\cos\frac{\gamma}{2}),$$
$$\eta^B_y = \cos\frac{\alpha}{2}\sin\frac{\tilde\alpha}{2}(\cos\frac{\beta}{2}\sin\frac{\tilde\gamma}{2}-\sin\frac{\beta}{2}\cos\frac{\tilde\gamma}{2})+\sin\frac{\alpha}{2}\cos\frac{\tilde\alpha}{2}(\cos\frac{\tilde\beta}{2}\sin\frac{\gamma}{2}-\sin\frac{\tilde\beta}{2}\cos\frac{\gamma}{2}),$$
$$\eta^B_z = \frac{1}{2}[\cos^2\frac{\alpha}{2}\cos\beta-\sin^2\frac{\alpha}{2}\cos\gamma+\cos^2\frac{\tilde\alpha}{2}\cos\tilde\beta-\sin^2\frac{\tilde\alpha}{2}\cos\tilde\gamma],$$
$$\delta^B_z = \frac{1}{2}[\cos^2\frac{\alpha}{2}\cos\beta-\sin^2\frac{\alpha}{2}\cos\gamma-\cos^2\frac{\tilde\alpha}{2}\cos\tilde\beta+\sin^2\frac{\tilde\alpha}{2}\cos\tilde\gamma].$$

With $\vec{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the two vectors, $\vec{r}^B$ and $\vec{r}$, are shown to be related by the vector transformation in Eq. (2.19). By introducing the denotations, $\gamma^A = \gamma$, $\tilde{\gamma}^A = \tilde{\gamma}$, $\gamma^B = \pi - \gamma$, and $\tilde{\gamma}^B = \pi - \tilde{\gamma}$, all the transformation elements derived here can be written into the compact form in Eq. (2.21).

2. Proof for Eq. (2.27)

According to the definition that $\eta_i = \frac{1}{2}(\eta_i^A + \eta_i^B)$ and $\delta_z = \frac{1}{2}(\delta_z^A + \delta_z^B)$, via Eq. (2.21), we have

$$\begin{aligned}
\eta_x &= \cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos(\frac{\pi}{4}-\frac{\beta}{2})\cos(\frac{\pi}{4}-\frac{\tilde{\gamma}}{2}) + \sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos(\frac{\pi}{4}-\frac{\tilde{\beta}}{2})\cos(\frac{\pi}{4}-\frac{\gamma}{2}),\\
\eta_y &= \cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos(\frac{\pi}{4}+\frac{\beta}{2})\cos(\frac{\pi}{4}-\frac{\tilde{\gamma}}{2}) + \sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos(\frac{\pi}{4}+\frac{\beta}{2})\cos(\frac{\pi}{4}-\frac{\gamma}{2}),\\
\eta_z &= \frac{1}{2}(\cos^2\frac{\alpha}{2}\cos\beta + \cos^2\frac{\tilde{\alpha}}{2}\cos\tilde{\beta}), \delta_z = \frac{1}{2}(\cos^2\frac{\alpha}{2}\cos\beta - \cos^2\frac{\tilde{\alpha}}{2}\cos\tilde{\beta}).
\end{aligned}$$

One may easily verify that, $\frac{\partial\eta_x}{\partial\gamma} \propto \sin(\frac{\pi}{4}-\frac{\gamma}{2})$, $\frac{\partial\eta_y}{\partial\gamma} \propto \sin(\frac{\pi}{4}-\frac{\gamma}{2})$, and $\frac{\partial\eta_z}{\partial\gamma} = \frac{\partial\delta}{\partial\gamma} = 0$. For the fidelity in Eq. (2.26), $F(\omega) = \frac{1}{2}(1 + \sum_i \eta_i\overline{n_i^2} + \delta_z\overline{n_z})$, there should be $\frac{\partial F}{\partial\gamma} \propto \sin(\frac{\pi}{4}-\frac{\gamma}{2})$. In a similar discussion, we find $\frac{\partial F}{\partial\tilde{\gamma}} \propto \sin(\frac{\pi}{4}-\frac{\tilde{\gamma}}{2})$. So, we can get $\gamma = \tilde{\gamma} = \frac{\pi}{2}$ by letting $\frac{\partial F}{\partial\gamma} = \frac{\partial F}{\partial\tilde{\gamma}} = 0$.

Appendix B: Optimal settings for QCMs

1. The case where θ is fixed.

From Eq. (2.31) and Eq. (3.1), we have the average fidelity, $\bar{F}^k = \frac{1}{2}(1 + \eta_\perp^k\sin^2\tilde{\theta} + \eta_z^k\cos^2\tilde{\theta} + \delta_z^k\cos\tilde{\theta})$, with $\eta_\perp^k$, η_z^k and δ_z^k given in Eq. (2.33). Using $\frac{\partial\bar{F}^A}{\partial\gamma} = -\frac{1}{4}\sin\frac{\alpha}{2}[\cos\frac{\tilde{\alpha}}{2}\sin\frac{\gamma}{2}\sin^2\tilde{\theta} + \sin\frac{\alpha}{2}\sin\gamma(\cos^2\tilde{\theta}+\cos\tilde{\theta})]$ and $\frac{\partial\bar{F}^B}{\partial\gamma} = \frac{1}{4}\sin\frac{\alpha}{2}[\cos\frac{\tilde{\alpha}}{2}\cos\frac{\gamma}{2}\sin^2\tilde{\theta} + \sin\frac{\alpha}{2}\sin\gamma(\cos^2\tilde{\theta}+\cos\tilde{\theta})]$, with $p\frac{\partial\bar{F}^A}{\partial\gamma} + (1-p)\frac{\partial\bar{F}^B}{\partial\gamma} = 0$, the equation get from Eq. (2.25), we arrive at $\sin\frac{\alpha}{2} = 0$. Putting it back into of fidelity in Eq.(2.31), we have $\bar{F}^A = \frac{1}{2}[1 + \sin\frac{\tilde{\alpha}}{2}\cos\frac{\tilde{\gamma}}{2}\sin^2\tilde{\theta} + \cos^2\tilde{\theta} + \sin^2\frac{\tilde{\alpha}}{2}\sin^2\frac{\tilde{\gamma}}{2}(\cos\tilde{\theta}-\cos^2\tilde{\theta})]$ and $\bar{F}^B = \frac{1}{2}[1+\sin\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\gamma}}{2}\sin^2\tilde{\theta}+\cos^2\tilde{\theta}+\sin^2\frac{\tilde{\alpha}}{2}\cos^2\frac{\tilde{\gamma}}{2}(\cos\tilde{\theta}-\cos^2\tilde{\theta})]$, where there should be $\frac{\partial\bar{F}^A}{\partial\tilde{\alpha}} = \frac{\cos\frac{\tilde{\alpha}}{2}}{4}[\cos\frac{\tilde{\gamma}}{2}\sin^2\tilde{\theta} + 2\sin\frac{\tilde{\alpha}}{2}\sin^2\frac{\tilde{\gamma}}{2}(\cos\tilde{\theta}-\cos^2\tilde{\theta})]$ and $\frac{\partial\bar{F}^B}{\partial\tilde{\alpha}} = \frac{\cos\frac{\tilde{\alpha}}{2}}{4}[\sin\frac{\tilde{\gamma}}{2}\sin^2\tilde{\theta} + 2\sin\frac{\tilde{\alpha}}{2}\cos^2\frac{\tilde{\gamma}}{2}(\cos\tilde{\theta}-\cos^2\tilde{\theta})]$. One may easily verify that the equation, $p\frac{\partial\bar{F}^A}{\partial\tilde{\alpha}} + (1-p)\frac{\partial\bar{F}^B}{\partial\tilde{\alpha}} = 0$, has a solution that $\cos\frac{\tilde{\alpha}}{2} = 0$. Finally, we find $\alpha = 0, \tilde{\alpha} = \pi$, the optimal setting which maximizes $\bar{F}^k$ if $\cos\tilde{\theta} \geq 0$.

2. Optimal setting for the phase-covariant cloning

For the phase-covariant case in Eq. (3.8), we also starts from Eq. (2.31) and get the average fidelities, $\bar{F}^A = \frac{1}{2}(1 + \sin\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos\frac{\gamma}{2} + \sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos\frac{\gamma}{2})$ and $\bar{F}^B = \frac{1}{2}(1 + \sin\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\sin\frac{\gamma}{2} + \sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\sin\frac{\gamma}{2})$. By requiring $\frac{\partial\bar{F}^A}{\partial\gamma}\frac{\partial\bar{F}^B}{\partial\tilde{\gamma}} - \frac{\partial\bar{F}^A}{\partial\tilde{\gamma}}\frac{\partial\bar{F}^B}{\partial\gamma} = 0$, we shall find the setting $\gamma = \tilde{\gamma}$ which simplifies the above average fidelities with $\bar{F}^A = \frac{1}{2}(1 + \cos\frac{\gamma}{2}\sin\frac{\alpha+\tilde{\alpha}}{2})$ and $\bar{F}^B = \frac{1}{2}(1 + \sin\frac{\gamma}{2}\sin\frac{\alpha+\tilde{\alpha}}{2})$. Finally, the result, $\alpha + \tilde{\alpha} = \pi$, can be easily achieved by asking $\frac{\partial\bar{F}^A}{\partial\alpha}\frac{\partial\bar{F}^B}{\partial\tilde{\alpha}} - \frac{\partial\bar{F}^A}{\partial\tilde{\alpha}}\frac{\partial\bar{F}^B}{\partial\alpha} = 0$.

3. Optimal proof for asymmetric universal cloning

For the special case where $\overline{n_i^2} = \frac{1}{3}$ while $\overline{n_i} = 0$, the fidelity in Eq. (2.31) should be,

$$\begin{aligned}
\bar{F}^A &= \frac{1}{2} + \frac{1}{3}(\cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos\frac{\tilde{\gamma}}{2} + \sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos\frac{\gamma}{2}) + \frac{1}{6}(\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}\cos\gamma + \cos^2\frac{\tilde{\alpha}}{2} + \sin^2\frac{\tilde{\alpha}}{2}\cos\tilde{\gamma}),\\
\bar{F}^B &= \frac{1}{2} + \frac{1}{3}(\cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\gamma}}{2} + \sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\sin\frac{\gamma}{2}) + \frac{1}{6}(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\cos\gamma + \cos^2\frac{\tilde{\alpha}}{2} - \sin^2\frac{\tilde{\alpha}}{2}\cos\tilde{\gamma}),
\end{aligned}$$

and a direct calculation shows, $\frac{\partial \bar{F}^A}{\partial \gamma}=-\frac{1}{6}\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}(\cos\frac{\tilde{\alpha}}{2}+2\sin\frac{\alpha}{2}\cos\frac{\gamma}{2})$, $\frac{\partial \bar{F}^A}{\partial \gamma}=\frac{1}{6}\cos\frac{\alpha}{2}\sin\frac{\gamma}{2}(\cos\frac{\tilde{\alpha}}{2}+2\sin\frac{\alpha}{2}\sin\frac{\gamma}{2})$, $\frac{\partial \bar{F}^A}{\partial \tilde{\gamma}}=-\frac{1}{6}\sin\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\gamma}}{2}(\cos\frac{\alpha}{2}+2\sin\frac{\tilde{\alpha}}{2}\cos\frac{\tilde{\gamma}}{2})$, $\frac{\partial \bar{F}^A}{\partial \tilde{\gamma}}=\frac{1}{6}\cos\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\gamma}}{2}(\cos\frac{\alpha}{2}+2\sin\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\gamma}}{2})$, $\frac{\partial \bar{F}^A}{\partial \tilde{\alpha}}=\frac{1}{6}(\cos\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos\frac{\tilde{\gamma}}{2}-\sin\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos\frac{\gamma}{2})-\frac{1}{6}\sin\tilde{\alpha}(1-\cos\tilde{\gamma})$, $\frac{\partial \bar{F}^B}{\partial \tilde{\alpha}}=\frac{1}{6}(\cos\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\gamma}}{2}-\sin\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\sin\frac{\gamma}{2})-\frac{1}{6}\sin\tilde{\alpha}(1+\cos\tilde{\gamma})$, $\frac{\partial \bar{F}^A}{\partial \alpha}=\frac{1}{6}(\cos\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos\frac{\gamma}{2}-\sin\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos\frac{\tilde{\gamma}}{2})-\frac{1}{6}\sin\alpha(1-\cos\gamma)$, and $\frac{\partial \bar{F}^B}{\partial \alpha}=\frac{1}{6}(\cos\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\sin\frac{\gamma}{2}-\sin\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\sin\frac{\tilde{\gamma}}{2})-\frac{1}{6}\sin\alpha(1+\cos\gamma)$. With $\frac{\partial \bar{F}^A}{\partial \gamma}\frac{\partial \bar{F}^B}{\partial \tilde{\gamma}}-\frac{\partial \bar{F}^A}{\partial \tilde{\gamma}}\frac{\partial \bar{F}^B}{\partial \gamma}=0$ and $\frac{\partial \bar{F}^A}{\partial \alpha}\frac{\partial \bar{F}^B}{\partial \tilde{\alpha}}-\frac{\partial \bar{F}^A}{\partial \tilde{\alpha}}\frac{\partial \bar{F}^B}{\partial \alpha}=0$, the two equations come from Eq. (2.25), we find $\tilde{\alpha}=\alpha$ and $\tilde{\gamma}=\gamma$ and arrive at the fidelities, $\bar{F}^A=\frac{1}{2}+\frac{2}{3}\sin\alpha\cos\frac{\gamma}{2}+\frac{1}{3}(\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}\cos\gamma)$ and $\bar{F}^B=\frac{1}{2}+\frac{2}{3}\sin\alpha\sin\frac{\gamma}{2}+\frac{1}{3}(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\cos\gamma)$, containing just two parameters, α and γ, here. Finally, from the two optimal equations, $p\frac{\partial \bar{F}^A}{\partial \gamma}+(1-p)\frac{\partial \bar{F}^B}{\partial \gamma}=0$ and $p\frac{\partial \bar{F}^A}{\partial \alpha}-(1-p)\frac{\partial \bar{F}^B}{\partial \alpha}=0$, and the average fidelity above, we may get the optimal settings of α and γ in Eq.(4.3).

4. Proof for Eq. (4.8)

By letting $\gamma=\tilde{\gamma}=0$ for Eq. (2.29), we get the fidelity, which is for the case defined in Eq. (4.6), with the expression that $F=\frac{1}{2}+\frac{1}{4}\cos^2\tilde{\theta}(\cos\frac{\alpha}{2}+\cos^2\frac{\tilde{\alpha}}{2})+\frac{\sqrt{2}}{4}\sin^2\tilde{\theta}(\cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}+\sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2})$. Requiring $\frac{\partial F}{\partial \alpha}=\frac{\partial F}{\partial \tilde{\alpha}}=0$, we shall get the optimal settings of $\tilde{\alpha}$ and α in Eq. (4.8).

5. Optimal setting for Eq. (5.2)

From Eq. (2.28) and Eq. (5.1), the fidelity should be $F=\frac{1}{2}\{1+(1-\overline{n_z^2})[\cos\frac{\alpha}{2}\sin\frac{\tilde{\alpha}}{2}\cos(\frac{\pi}{4}-\frac{\beta}{2})+\sin\frac{\alpha}{2}\cos\frac{\tilde{\alpha}}{2}\cos(\frac{\pi}{4}-\frac{\tilde{\beta}}{2})]+\frac{1}{2}\overline{n_z}^2(\cos^2\frac{\alpha}{2}\cos\beta+\cos^2\frac{\tilde{\alpha}}{2}\cos\tilde{\beta})+\frac{1}{2}\overline{n_z}(\cos^2\frac{\alpha}{2}\cos\beta-\cos^2\frac{\tilde{\alpha}}{2}\cos\tilde{\beta})\}$. The setting $\tilde{\alpha}=\pi$, which is get from the result $\partial F/\partial\tilde{\beta}\propto\cos\frac{\tilde{\alpha}}{2}$, simplifies the fidelity with $F=\frac{1}{2}\{1+(1-\overline{n_z^2})\cos\frac{\alpha}{2}\cos(\frac{\pi}{4}-\frac{\beta}{2})+\frac{1}{2}(\overline{n_z}^2+\overline{n_z})\cos^2\frac{\alpha}{2}\cos\beta\}$. The setting $\alpha=0$ holds since $\frac{\partial F}{\partial\alpha}\propto\sin\frac{\alpha}{2}$. The optimal setting of β in Eq. (5.2), can be directly calculated from the equation $\partial F/\partial\beta=0$ with $F=\frac{1}{2}\{1+(1-\overline{n_z^2})\cos(\frac{\pi}{4}-\frac{\beta}{2})+\frac{1}{2}(\overline{n_z}^2+\overline{n_z})\cos\beta\}$.

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